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Stability and non-standard finite difference method of the generalized Chua's circuit

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ABSTRACT

In this paper, we develop a framework to obtain approximate numerical solutions of the fractional-order Chua's circuit with Memristor using a non-standard finite difference method. Chaotic response is obtained with fractional-order elements as well as integer-order elements. Stability analysis and the condition of oscillation for the integer-order system are discussed. In addition, the stability analyses for different fractional-order cases are investigated showing a great sensitivity to small order changes indicating the poles' locations inside the physical *s*-plane. The Grünwald–Letnikov method is used to approximate the fractional derivatives. Numerical results are presented graphically and reveal that the non-standard finite difference scheme is an effective and convenient method to solve fractional-order chaotic systems, and to validate their stability.

1. Introduction

The Memristor is the fourth fundamental electrical element which provides a relationship between flux φ and charge q, where $d\varphi = Mdq$. This element was described in 1971 by Chua [1]. The first realization of a real working passive Memristor was demonstrated by HP Labs in 2008 [2]. Although the Memristor has mathematically a scalar state, its system has vector state due to its memory property. There are a wide range of promising applications of the Memristor's ability to realize new analog computers similar to the human brain [1].

Recently, the characterization of real dynamical systems using fractional-order dynamical models has proved to be superior to the traditional calculus [3]. Many generalized fundamentals were extracted and were reduced to their known responses when the fractional orders converges to integer values. Fractional derivatives provide an excellent instrument to describe memory and hereditary properties of various materials and processes. Fractional differentiation and integration operators are used to model problems in astrophysics [4–7], chemical physics, signal processing, systems identification, control and robotics [8,9], and many other areas.

The event, which led to the discovery the first chaotic electronic circuit (Chua's circuit shown in Fig. 1(b)), was an accident in Matsumoto's lab while the theoretical study of chaos was known since a long time ago. Similarly, the first passive realization of the Memristor was invented by chance in the HP lab shown in Fig. 1(a). In addition, the realization of fractional-order elements will sooner or later be available for commercial needs since a silicon based half-order capacitor [10] already exists.

However, one of the circuit equivalents of the general fractional-order capacitor [11,12] is shown in Fig. 1(c), where the values of $\{R_k, C_k : k = 1, 2, \dots, N\}$ are optimized to achieve a certain error for each fractional-order parameter in [12]. Many

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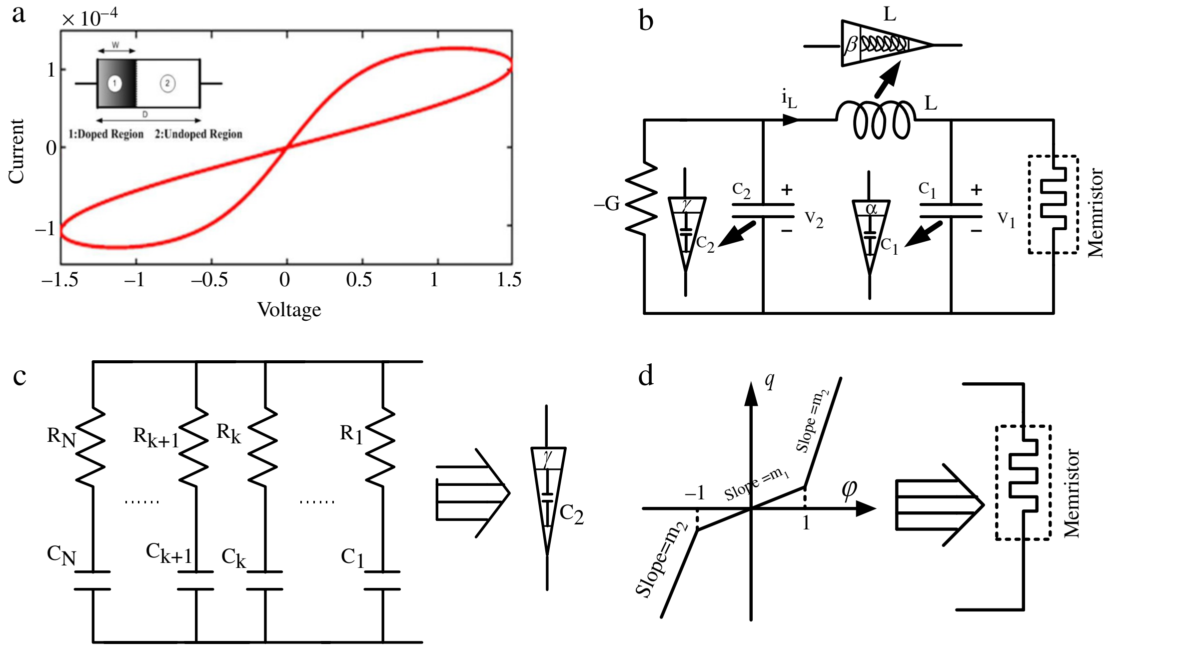

Fig. 1. (a) The invented HP Memristor and its I–V characteristic, (b) the canonical Chua's circuit with a flux-controlled Memristor, (c) general realization of the fractional-order capacitor, and (d) the constitutive relation of the Chua's monotone-increasing piecewise-linear flux-controlled Memristor.

generalized theorems are investigated based on this element as in [11,13]. In addition, there are many types of Memristors, such as the first proposed Chua's models [1], up to the first passive HP Memristor [2]. Moreover, the Chua's Memristor models are divided into flux-controlled and charge-controlled piecewise-linear Memristors as shown in Fig. 1(d). This Memristor has two different slopes m_1, and m_2 according to the flux across the Memristor. Many mathematical models for the Memristor are presented to have the same I–V hysteresis as in [14].

Finding robust and stable numerical and analytical methods for solving the fractional differential equations has been an active research undertaken several authors. These methods include the Adomian decomposition and variational iterational methods [15,16]. The Adams–Bashforth–Moulton method is one of the most used methods to solve fractional differential equations [17]. Recently, the non-standard finite difference method (NSFD) has been applied for the numerical solutions of fractional differential equations and the results show that the NSFD leads to faster convergence and more accurate results when compared by standard alternative methods [18]. Erjaee [19] investigated the saddle and Hop bifurcation points of predator–prey fractional differential equations system with a constant rate harvesting using the NSFD.

In this paper, we study the effect of both Memristor's properties and fractional (non-integer) elements instead of the conventional resistor, and storage elements (L or C) on the Chua's circuit. The generalized system will be fractional-order nonlinear differential equations with memory, where its solution depends on non-standard finite difference method. This paper is devoted to study the stability analysis and to develop a non-standard discretization scheme given by Mickens [20] to the Grünwald–Letnikov discretization process for the linear and nonlinear fractional differential systems. The non-standard finite difference (NSFD) scheme Mickens [21–23] has been developed as an alternative method for solving a wide range of problems whose mathematical models involve algebraic, differential, biological models, and chaotic systems. The definition of Grünwald–Letnikov derivative has been used in numerical analysis to discretize the fractional differential equations with Riemann–Liouville derivative. The technique has many advantages over the classical techniques, and provides an efficient numerical solution.

The rest of the paper is organized as follows. In the next section, the conventional Chua's circuit and its state-space presentation, followed by stability analysis in the integer-order case indicating the necessary condition of oscillation. Then, the stability analysis of the fractional-order system is discussed for different order and variables showing a huge sensitivity with respect to small change in fractional order. Section 4, presents the mathematical preliminaries and the NSFD scheme to solve the fractional differential systems. In Section 5, we present different numerical approximations to the solutions for different fractional-order cases. In the last section, we summarize the conclusions.

2. Memristor-based Chua's circuit

Fig. 1(b) shows the canonical Memristor-based integer-order Chua's circuit. It was verified in [24], that this circuit could behave as a chaotic generator. The state-variables of this circuit can be transformed into dimensionless form by defining the

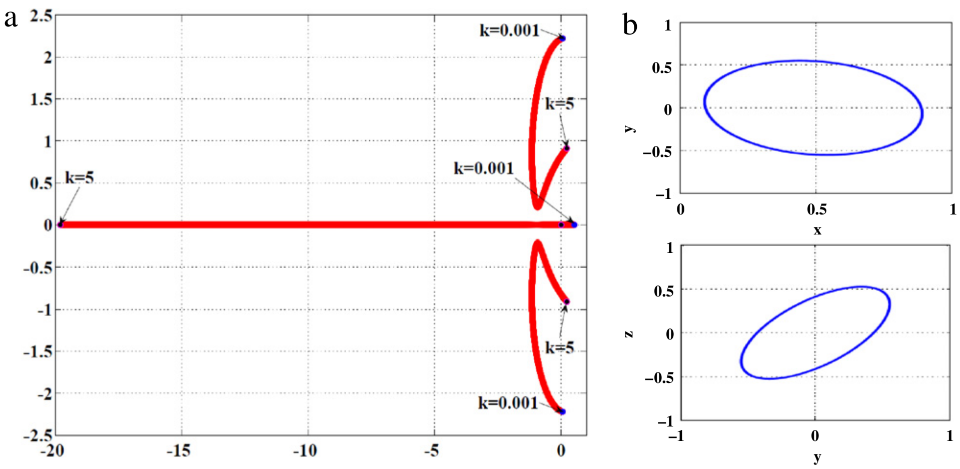

Fig. 2. (a) Poles in the s-plane for k [0.001, 5], and (b) periodic orbits for the x–y, and y–z when $k = 1.6634$, in the integer case (1.0, 1.0, 1.0).

variables $x = v_1,\ y = i_3,\ z = v_3,\ u = \varphi, L = 1H,\ a = 1/C_1,\ b = 1/C_2,\ c = G/C_2$. Then, the system of equations will be

$$\dot{P} = JP, \qquad \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{u} \end{pmatrix} = \begin{pmatrix} -ak & a & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -b & c & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix}, \qquad k = f(u) = \frac{\mathrm{d}q(u)}{\mathrm{d}u} = \begin{cases} m_1 & |u| < 1 \\ m_2 & |u| > 1 \end{cases} \tag{1}$$

where a, b, c, m_1, and m_2 are parameters. The effect of the memristive property appears through the function $q(u)$, which is a piecewise-linear function with two different slopes m_1 and m_2 depending on u.

3. Stability analysis

The stability analysis of the differential equations is very important according to the required application's behavior. Each behavior is related to the locations of the system's poles with respect to the equilibrium points. The only equilibrium of this system is $\hat{P} = (\hat{x},\ \hat{y},\ \hat{z},\ \hat{u}) = (0,\ 0,\ 0,\ 0)$ due to the charge–flux nonlinear relationship of the Memristor. Two individual regions of study are studied to analyze the solution of this system. The first region when $k = m_1$ and the second when $k = m_2$. In this section, we will study the eigenvalue problem of the integer-order and fractional-order cases for different values of k.

3.1. Integer-order system $(\alpha,\ \beta,\ \gamma) = (1, 1, 1)$

The characteristic polynomial of (1) evaluated at the equilibrium point P is given by:

$$\lambda^4 + (ak - c)\lambda^3 + (b - ack + a)\lambda^2 + a(bk - c)\lambda = 0. \tag{2}$$

One of these eigenvalues is always zero, however the other three values are dependent on the circuit parameters. Through (2), the system will oscillate if the value of k satisfies the following equation

$$k^2 - \left(\frac{1}{c} + \frac{c}{a}\right) k + \frac{b}{a^2} = 0. \tag{3}$$

For example, when $a = 4$, $b = 1$, and $c = 0.65$ the previous quadratic equation (2) has two solutions at $k = 0.037$ and $k = 1.6634$ where the eigenvalues are $\{0.5, \pm 2.214j\}$ and $\{-6, \pm 8.217j\}$ respectively. Since the first solution is unstable due to the positive eigenvalues, the system will oscillate only when $k = 1.6634$. Note that, if $k =$ constant then the Memristor will return to the conventional meaning as resistor. Fig. 2(a) shows the poles locations versus k, when k spans the range [0.001, 5]. From this figure, the poles at $k = 0.001$ are $\{0.526, 0.06 \pm 2.22j\}$, which are located in the right-half plane producing unstable performance. However as k increases, the real pole goes into the negative direction entering the left-half plane. When $k = 5$, the real pole is highly negative -19.8 with respect to the other two conjugate poles $\{0.224 \pm 0.91j\}$. Since the sum of them is still negative, then the system can exhibit a chaotic performance. It is clear that, the poles curves will cross the $j\omega$ axis at two cases of k as mentioned before. Fig. 2(b) shows the periodic response of the x–y trajectory when $k = 1.6634$ as expected. Also, if $k = c/b = 0.65$, the real pole will have another zero eigenvalue. Therefore, if $k < 0.65$ or $k > 1.6634$, the system will have a pole at least in the RHP. However, if $0.65 < k < 1.6634$, the system will always be stable.

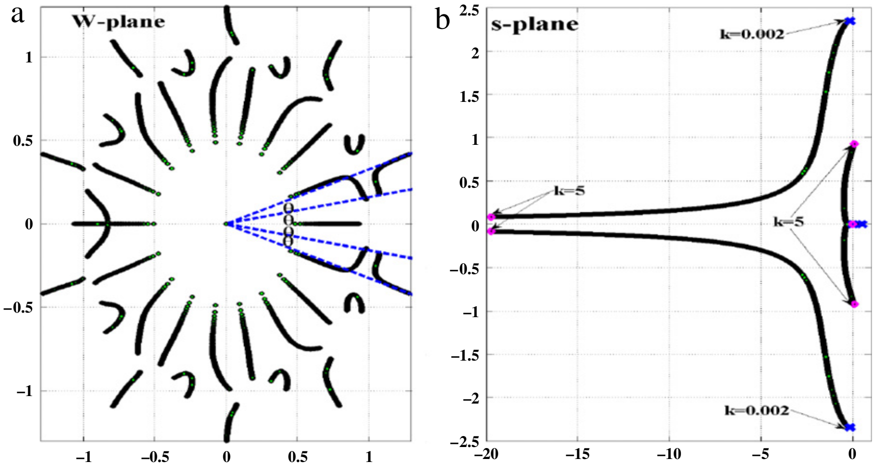

Fig. 3. Poles in (a) W-plane and (b) s-plane for the integer case (1.0, 0.9, 0.9).

3.2. Fractional-order system analysis

Two planes are shown in the fractional-order case where the first is the so-called W-plane which has all poles either in physical or nonphysical planes. After transformation into the s-plane, the physical poles for different values of k in the s-plane are also introduced. By replacing the conventional capacitors and inductors by fractional ones with orders (α, β, γ) as shown in Fig. 1(b), the new Memristor-based fractional-order Chua's system will be

$$D^{\alpha}x(t) = a(y - f(u)x) = a(y - kx), \tag{4a}$$

$$D^{\beta}y(t) = z - x, \tag{4b}$$

$$D^{\gamma}z(t) = -by + cz, \tag{4c}$$

$$Du(t) = x. \tag{4d}$$

One of the methods to study the stability of fractional-order systems is the W-plane method [25], which can be used for different fractional orders inside the same system. The Laplace s-plane of the system in the fractional-order domain is given by

$$s^{\alpha}X(s) = aY(s) - kX(s), \qquad s^{\beta}Y(s) = Z(s) - X(s), \qquad s^{\gamma}Z(s) = -bY(s) + cZ(s), \qquad sU(s) = X(s). \tag{5}$$

Therefore, the general characteristic in the s-plane is given by

$$s^{\alpha+\beta+\gamma} - cs^{\alpha+\beta} + aks^{\beta+\gamma} - caks^{\beta} + as^{\gamma} + bs^{\alpha} + (abk - ac) = 0. \tag{6}$$

Example: when $\alpha = 1.0$, $\beta = 0.9$, and $\gamma = 0.9$. Let us define $W = s^{1/m} = s^{0.1}$, then the above nonlinear equation will be a polynomial equation in W as follows

$$W^{28} - cW^{19} + akW^{18} - a(ck - 1)W^{9} + bW^{10} + (abk - ac) = 0. \tag{7}$$

There are 28 roots for each value of k, some of them can be retransformed back into the s-plane, which is called the physical plane, and the others can not. Only the roots whose angles satisfy (8) can exist in the physical s-plane [25]. When $m = 10$, the s-plane is represented by tenth the W-plane as shown from Fig. 3(a) from the outer dotted lines, while the inner dotted lines represent the equivalent $\pm j\omega$ axis in the s-plane. Thus any poles between the inner and outer dotted lines are equivalent to the left-half of the s-plane (stable poles). Fig. 3(a) illustrates all possible roots in the W-plane in case when $k \in [0.002, 5]$ with step 0.001. However, Fig. 3(b) shows only the poles in the physical s-plane through the transformation $s = W^{10}$ for poles between the outer dotted lines in Fig. 3(a).

$$|\angle W| \le \frac{\pi}{m} = 2\theta. \tag{8}$$

From Fig. 3(b), it is clear that the response at $k = 0.002$ is still unstable due to the locations of the poles in the s-plane. However, when $k = 5$, the system has two pairs of conjugate poles, where two of them have highly negative value which can be used for chaotic behavior. Similarly, Figs. 4 and 5 discuss the poles in both W and s planes for the same range of k but with slight changes in the fractional-order α. The poles' locations are totally changed in both planes. In Fig. 4(b) where

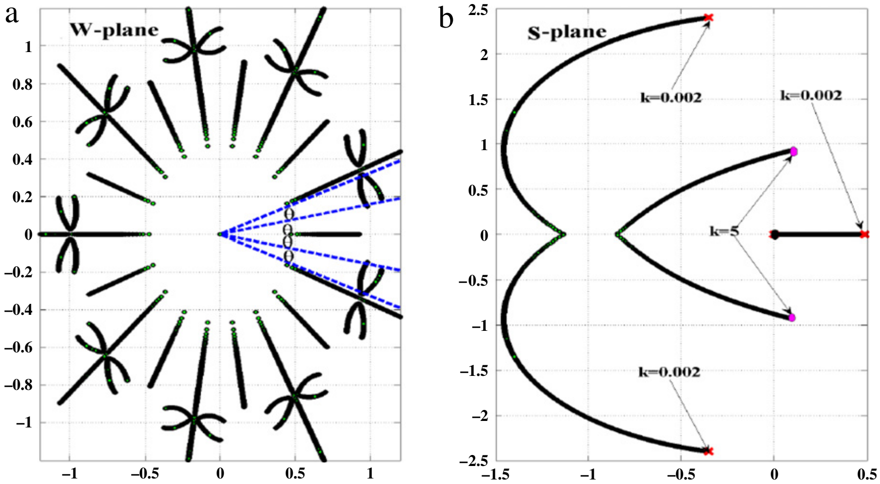

Fig. 4. Poles in (a) W-plane and (b) s-plane for the integer case (0.9, 0.9, 0.9).

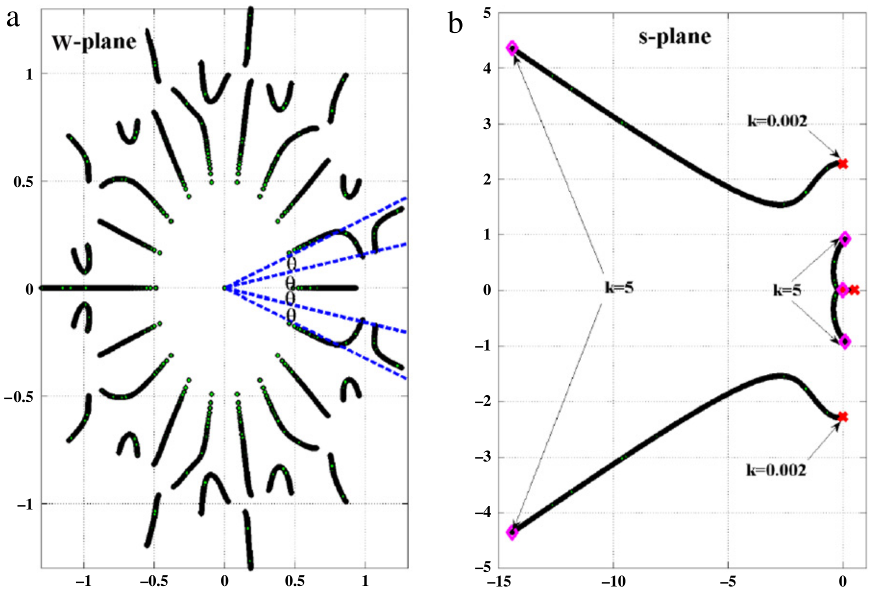

Fig. 5. Poles in (a) W-plane and (b) s-plane for the integer case (1.1, 0.9, 0.9).

$(\alpha, \beta, \gamma) = (0.9, 0.9, 0.9)$, the system will be unstable for $k = 5$ since there are only two poles inside the physical s-plane with a positive real part. However, at $k = 0.002$, there are three roots with negative sum. The details of poles' locations for different fractional orders and k values are listed in Table 1.

When $(\alpha, \beta, \gamma) = (1.1, 0.9, 0.9)$, the poles' location in both W and s planes are plotted in Fig. 5(b), where the system is unstable at $k = 0.002$ due the positive sum of these roots which is unlike the big negative sum with four roots in the case when $k = 5$.

Conversely, the small changes in one fractional order, huge poles and their numbers are different depending on these changes and may affect the stability at the same points as shown from Figs. 3–5.

4. Non-standard finite difference method for a fractional system

In this section, we give some basic definitions and properties of the fractional calculus theory and non-standard discretization, which are used further in this paper.

Table 1
Poles of fractional-order Chua's system for different orders and k values.

Case	(α, β, γ)	k	λ	$\Sigma\lambda$
1	(1.0, 1.0, 1.0)	0.001	$0.526, 0.06 \pm 2.22j$	1.12
2		0.037	$0.5, 0.0026 \pm 2.21j$	1.0026
3		1.6634	$-6, \pm 8.217j$	-6
4		5	$-19.8, 0.224 \pm 0.91j$	-20.248
5	(0.9, 0.9, 0.9)	0.002	$0.489, -0.351 \pm 2.4j$	-0.213
6		0.2	$0.3454, -0.724 \pm 2.25j$	-1.1026
7		5	$0.087 \pm 0.927j$	0.174
8	(1.0, 0.9, 0.9)	0.002	$0.496, -0.162 \pm 2.343j$	0.172
9		0.2	$0.3523, -0.503 \pm 2.28j$	-0.6537
10		5	$-19.74 \pm 0.084j$ $-0.0864 \pm 0.926j$	-39.6528
11	(1.1, 0.9, 0.9)	0.002	$0.5022, -0.0036 \pm 2.276j$	0.495
12		0.2	$0.3586, -0.304 \pm 2.28j$	-0.2494
13		5	$-14.384 \pm 4.36j$ $0.0857 \pm 0.926j$	-28.5966

4.1. Grünwald –Letinkov approximation

We will begin with the single fractional differential equation, (see [18])

$$D^{\alpha}x(t) = f(t, x(t)), \quad T \geq t \geq 0, \quad \text{and} \quad x(t_0) = x_0, \tag{9}$$

where $\alpha > 0$ and D^{α} denotes the fractional derivative, defined by

$$D^{\alpha}x(t) = J^{n-\alpha}D^{n}x(t), \tag{10}$$

where $n - 1 < \alpha \leq n$, $n \in N$ and J^n in the nth-order Riemann–Liouville integral operator defined as

$$J^{n}\,x(t) = \frac{1}{\Gamma(n)}\int_{a}^{t}(t-\tau)^{n-1}\,x(\tau)\,\mathrm{d}\tau, \quad t > 0. \tag{11}$$

To apply Mickens' scheme, we have chosen this Grünwald–Letnikov method approximation for the one-dimensional fractional derivative as follows:

$$D^{\alpha}x(t) = \lim_{h\to 0} h^{-\alpha}\sum_{j=0}^{N}(-1)^{j}\binom{\alpha}{j}x(t - jh), \tag{12}$$

where $N = [t/h]$. Usually, we use $[t]$ to denote the integer part of t and h is the step size. Therefore, Eq. (9) is discretized as follow

$$\sum_{j=0}^{n+1} c_j^{\alpha}\,x(t - jh) = f(t_n, x(t_n)), \quad n = 1, 2, 3, \ldots, \tag{13}$$

where $t_n = nh$ and c_j^{α} are the Grünwald–Letnikov coefficients defined as

$$c_j^{\alpha} = \left(1 - \frac{1+\alpha}{j}\right)c_{j-1}^{\alpha}, \quad j = 1, 2, 3, \ldots, \text{where } c_0^{\alpha} = h^{-\alpha}. \tag{14}$$

4.2. Non-standard discretization

In general, the non-standard finite difference rules, introduced by Mickens [20], do not lead to a unique discrete model for either ODEs. Therefore, we give the basic rules of non-standard ODEs. We seek to obtain the NSFD solution for a system of differential equations of the form

$$x_k' = f(t, x_1, x_2, \ldots, x_m), \quad k = 1, 2, \ldots, m. \tag{15}$$

Using the finite difference method, the discrete derivatives are

$$x_1' = \frac{x_{1,k+1} - x_{1,k}}{\varphi_1(h)}, \qquad x_2' = \frac{x_{2,k+1} - x_{2,k}}{\varphi_2(h)}, \quad \ldots \qquad x_m' = \frac{x_{m,k+1} - x_{m,k}}{\varphi_m(h)}, \tag{16}$$

where $\varphi_k(h)$ are functions of the step size $h = \Delta t$, and have the following properties:

$$\varphi_k(h) = h + O(h^2), \quad \text{where } h \to 0. \tag{17}$$

Examples of functions $\varphi_k(h)$ that satisfy Eq. (17) are: $h, \sin h, \ \sinh h, \ 1 - e^{-h}$. The nonlinear terms can be in general being replaced by nonlocal discrete representations. For example, $x^2 = x_k x_{k+1}$ where $h = T/N, \ t_n = nh, n = 0, 1, \ldots, N \in Z^+$. Now, we apply the NSFD to obtain the numerical solution for the fractional-order Chua's circuit system. Using the Grünwald–Letnikov discretization method and applying Mickens scheme by replacing the step size h by a function $\varphi(h)$ the system (4) yields

$$\sum_{j=0}^{k+1} c_j^{\alpha} u(t_{k+1-j}) = a(y(t_k) - f(x(t_{k+1}))u(t_{k+1})), \quad \text{where } x(t_0) = x_0, \ u(t_0) = u_0, \tag{18a}$$

$$\sum_{j=0}^{k+1} c_j^{\beta} y(t_{k+1-j}) = z(t_k) - u(t_{k+1}), \quad y(t_0) = y_0, \tag{18b}$$

$$\sum_{j=0}^{k+1} c_j^{\gamma} z(t_{k+1-j}) = -by(t_{k+1}) + cz(t_k), \quad z(t_0) = z_0, \tag{18c}$$

after doing some algebraic manipulation Eqs. (18) give

$$x_{k+1} = \frac{-\sum_{j=1}^{k+1} c_j^{\alpha} x(t_{k+1-j}) + ay(t_k)}{c_0^{\alpha} + af(u(t_{k+1}))}, \tag{19a}$$

$$y_{k+1} = \frac{-\sum_{j=1}^{k+1} c_j^{\beta} y(t_{k+1-j}) + z(t_k) - x(t_{k+1})}{c_0^{\beta}}, \tag{19b}$$

$$z_{k+1} = \frac{-\sum_{j=1}^{k+1} c_j^{\gamma} z(t_{k+1-j}) - by(t_{k+1})}{c_0^{\gamma} - c}, \tag{19c}$$

$$u_{k+1} = u(t_k) + hx(t_{k+1}), \tag{19d}$$

where $c_0^m = \varphi(h)^{-m}, m = \alpha, \beta, \gamma$. To obtain a reasonable value of the function $\varphi(h)$, we choose it in the exponential form [26]

$$\varphi(h) = 1 - e^{-h}. \tag{20}$$

5. Results and discussions

In all the calculations done in this paper, the fractional system was numerically integrated using the NSFD scheme. We chose the time step size $h = 0.001$ and the parameters $a = 4, \ b = 1, \ c = 0.65, \ d = 1, m_1 = 0.2, \ m_2 = 5$. Due to the existence of new fractional orders $(\alpha, \ \beta, \ \gamma)$, three more degrees of freedom are available. These extra parameters can be used as control variables, best fit to a certain behavior through optimization techniques, and expand the chaotic range. For example, if this integer-order circuit was verified to behave as a chaotic oscillator at any given parameters of $(m_1, m_2, a, b, c,$ and $d)$, we can study the chaotic response for different values of $(\alpha, \ \beta, \ \gamma)$ around the integer orders (1, 1, 1) that exhibit chaotic behavior. We will discuss three different cases using numerical and circuit simulation cases.

5.1. Numerical time waveforms

The sensitivity of the chaotic behavior with respect to the fractional-order parameters are shown in Fig. 6, where two different cases are studied with the same $\beta = \gamma = 0.9$, and slightly small changes in the fractional order $\alpha = 1.0$, and 1.1 respectively. Although, the waveforms are similar, the number of peaks and their values are different. Therefore, the long-term memory associated with the fractional order $\alpha = 1.1$ plays the main role in the differences between the two figures. Similarly, we can discuss the sensitivity of the output time waveforms with other fractional-order parameters. Three time-domain waveforms for the variables $u, \ y, \ z$ in these two different cases are illustrated in Fig. 6(a) and (b) respectively.

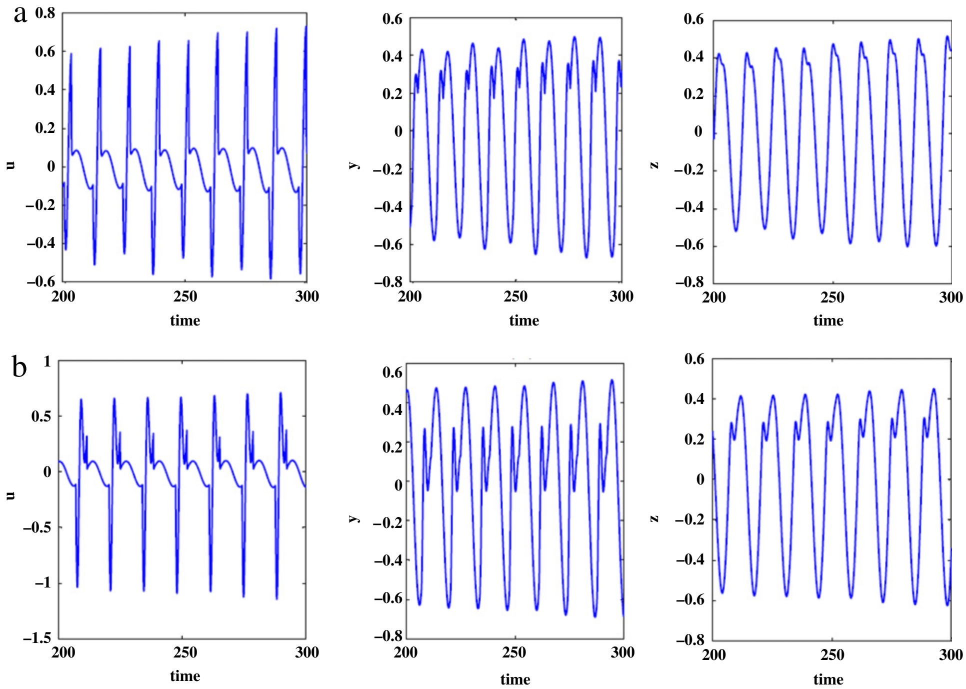

Fig. 6. Time waveforms for u, y, and z for (a) (1.0, 0.9, 0.9), and (b) (1.1, 0.9, 0.9).

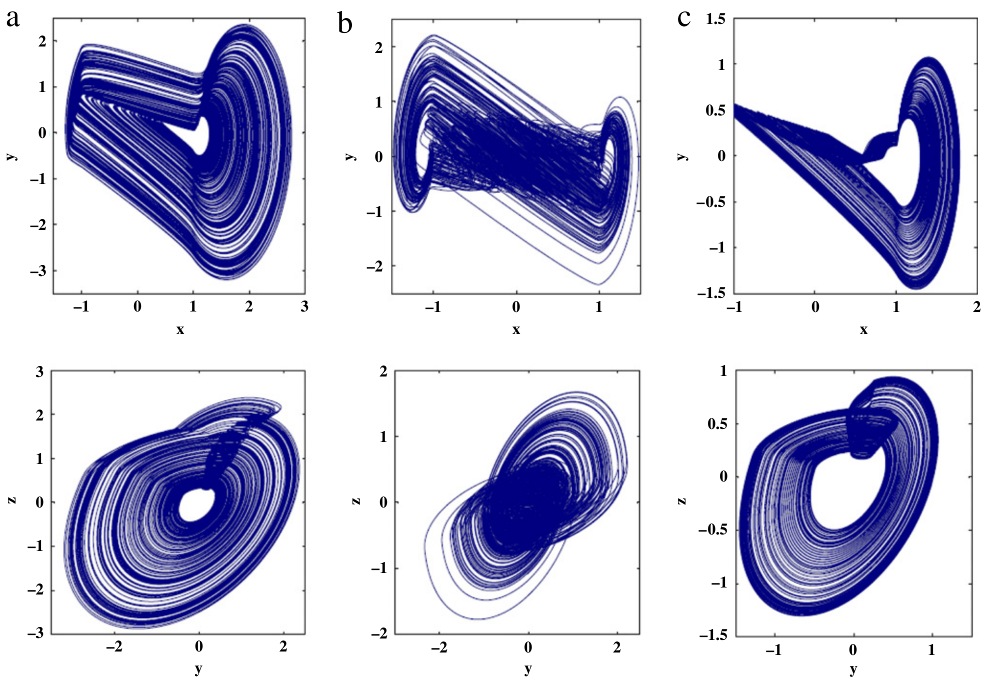

Fig. 7. Attractors x–y, y–z for (α, β, γ) (a) (1.0, 0.9, 0.9), (b) (1.0, 1.0, 1.0), and (c) (1.1, 0.9, 0.9).

5.2. Shape of the strange attractors

One of the main properties, which distinguish between chaotic systems, is the shape of strange attractors they display. The effect of fractional order in the previous system can be extended not only to slight changes in the time waveforms as we discussed before, but it can change the whole attractor shape as shown in Fig. 7. Two different strange attractors for three distinct fractional-order systems are illustrated. Using same initial conditions and the same number of points and comparing

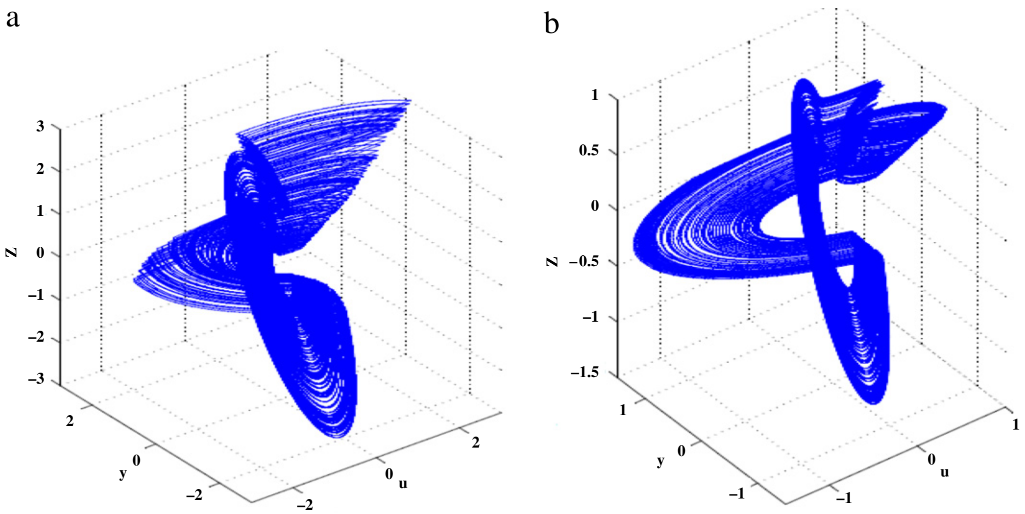

Fig. 8. 3D shapes in the u–y–z plane when $(\alpha,\ \beta,\ \gamma)$ (a) (1.0, 0.9, 0.9), and (b) (1.1, 0.9, 0.9).

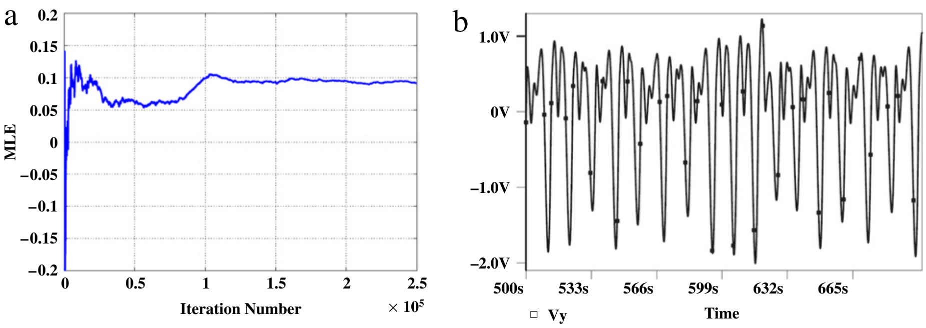

Fig. 9. (a) Maximum Lyapunov exponent, and (b) time waveform of V_{C2} (y variable).

Fig. 7(a) with (c), the attractors' shape are highly changed due to the long-term memory of the fractional-order derivatives. The difference in the number of peaks and their values appears clearly.

The ranges of variables are highly affected for example, z parameter ranges are $(-3, 3)$ and $(-1.5, 1)$ when $(\alpha,\ \beta,\ \gamma)$ are from (0.9, 0.9, 0.9) and (1.1, 0.9, 0.9) respectively. In addition, the 3D shapes in the space u–y–z are plotted in Fig. 8. for the two cases when $(\alpha,\ \beta,\ \gamma) = (1.0, 0.9, 0.9)$, and (1.1, 0.9, 0.9) respectively, where the effect of the "u" variable exists in these graphs. As a verification that the response is chaotic, the maximum Lyapunov exponent (MLE) is calculated using the software provided in [27] which is based on the nonlinear time-series analysis. The time evolution of the MLE for x variable illustrates saturation to positive value of approximately 0.0915 over the long term as shown Fig. 9(a) based on 250,000 points.

5.3. Circuit simulation

By using an integer-order inductor and fractional-order capacitor with order 0.9 as discussed in [13], which is based on integer-capacitors with resistors, a good approximation can be obtained with good finite memory. As the number of stages increases (N), the more we obtain good approximation The Chua's piecewise flux-controlled Memristor is built using ideal realization. Fig. 9(b), shows the time waveforms of the voltage across C_2 (y variable) under the same conditions.

6. Conclusion

NFDM is used to obtain the attractor for chaotic system in Memristor-based fractional-order Chua's circuit, where chaos is verified for different fractional-order capacitors and inductor.

Stability study in both the W-plane and s-plane are discussed for general different orders showing the physical s-plane and the equivalent poles' locations. Several cases of the fractional derivative are introduced with stability analysis. Chaotic behavior is verified from the calculation of the MLE and from the circuit simulations. The results presented in this paper suggested that this algorithm is also readily applicable to more complex chaotic systems such as Chen and hyperchaotic systems.

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